

Domains of Attraction of Multivariate Extreme Value Distributions

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Some necessary and sufficient conditions for domains of attraction of multivariate extreme value distributions are shown by using dependence functions. The joint asymptotic distribution of multivariate extreme statistics is also shown.

Key words: asymptotic joint distribution; dependence function; multivariate extreme statistics.

1. Introduction

Multivariate extreme value distributions have been studied by many authors, and their contributions are summarized by Galambos [1] and Resnick [2]. The purpose of this paper is to obtain some necessary and sufficient conditions for domains of attraction of the multivariate extreme value distributions. The joint asymptotic distribution of multivariate extreme statistics is also obtained. To study multivariate extreme value distributions and their domains of attraction, Sibuya [3] introduces the notion of a dependence function which is also used by Galambos [1]. A dependence function or copula is a useful notion to construct a family of joint distributions.

In this paper, basic arithmetical operations are always meant componentwise (see Galambos [1], Chapt. 5).

Let $(X_{1j}, X_{2j}, \dots, X_{kj})$, $j=1, 2, \dots, n$, be a sample of size n , of a k -dimensional random vector with a distribution function $F(\mathbf{x})$. The i -dimensional distribution function of the components $X_{j1}, X_{j2}, \dots, X_{ji}$ will be denoted $F_{j1j2\dots ji}(x_{j1}, x_{j2}, \dots, x_{ji}) = F_{J(i)}(\mathbf{x}_{J(i)})$. We shall also use the notation $\bar{F}_{j1\dots ji}(x_{j1}, \dots, x_{ji}) = \bar{F}_{J(i)}(\mathbf{x}_{J(i)}) = P(X_{j1} > x_{j1}, \dots, X_{ji} > x_{ji})$. For $k=1$ and $p \in (0, 1)$, let $F^{-1}(p) = \inf\{x : F(x) \geq p\}$.

Let $\mathbf{Z}_n = (Z_{1n}, \dots, Z_{kn})$, where $Z_{in} = \max\{X_{i1}, \dots, X_{in}\}$, $i=1, 2, \dots, k$, and let us call \mathbf{Z}_n a multivariate extreme statistic.

If there exist $\mathbf{a}_n > \mathbf{0}$, $\mathbf{b}_n \in R^k$, $n=1, 2, \dots$ ($\mathbf{a}_n > \mathbf{0}$ means $a_{in} > 0$, $i=1, \dots, k$) such that $(\mathbf{Z}_n - \mathbf{b}_n)/\mathbf{a}_n$ converges in distribution to a random vector \mathbf{U} with a nondegenerate distribution H (i.e., all univariate marginals of H are nondegenerate), then F is said to be in the domain of attraction of H , $F \in D(H)$ by symbol, and H is said to be a multivariate extreme value distribution. The convergence in distribution is equivalent to the condition

$$\lim_{n \rightarrow \infty} F^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) = H(\mathbf{x}) \quad (1)$$

for all \mathbf{x} , because multivariate extreme value distributions are continuous.

We shall need the following lemma to prove a proposition in Sec. 2.

Lemma 1.1 Equation (1) is equivalent to

$$\lim_{n \rightarrow \infty} n[1 - F(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n)] = -\log H(\mathbf{x})$$

for all \mathbf{x} such that $0 < H(\mathbf{x}) < 1$. (See Marshall and Olkin [4].)

2. Domains of Attraction

For any k -dimensional distribution F ,

$$D_F(\mathbf{y}) = F(F_1^{-1}(y_1), \dots, F_k^{-1}(y_k)), \mathbf{y} = (y_1, \dots, y_k) \in (0, 1)^k$$

is called the dependence function of F . In this section, we derive necessary and sufficient conditions for domains of attraction in terms of the dependence function.

Proposition 2.1 Let F be a k -dimensional distribution and let H be a multivariate extreme value distribution with univariate marginals H_i , $i = 1, \dots, k$. Then the following statements are equivalent:

- 1) $F \in D(H)$.
- 2) $F_i \in D(H_i)$, $i = 1, \dots, k$, and

$$\lim_{s \rightarrow \infty} s[1 - D_F(\mathbf{y}^{1/s})] = -\log D_H(\mathbf{y}) \text{ for all } \mathbf{y} \in (0, 1)^k.$$

- 3) $F_i \in D(H_i)$, $i = 1, \dots, k$, and

$$\lim_{x \uparrow 1} \frac{1 - D_F(\mathbf{y}^{1-x})}{1-x} = -\log D_H(\mathbf{y}) \text{ for all } \mathbf{y} \in (0, 1)^k.$$

- 4) $F_i \in D(H_i)$, $i = 1, \dots, k$, and

$$\lim_{y \downarrow 0} \frac{1 - D_F(\mathbf{y}')}{1 - D_H(\mathbf{y}')} = 1 \text{ for all } \mathbf{y}' \in (0, 1)^k.$$

Proof. The proof is straightforward from Lemma 1.1, Theorem 5.2.3 and Lemma 5.4.1 of Galambos [1]. \square

Proposition 2.2 Let F be a k -dimensional distribution and let H be a multivariate extreme value distribution with univariate marginals H_i , $i = 1, \dots, k$.

(A) $F \in D(H)$ if and only if $F_i \in D(H_i)$, $i = 1, \dots, k$, and the functions

$$d_{J(i)}(\mathbf{y}_{J(i)}) = \lim_{n \rightarrow \infty} n \bar{D}_{F_{J(i)}}(\mathbf{y}_{J(i)})^{1/n}$$

for each fixed vector $J(i)$ ($i > 1$) and for all $\mathbf{y} \in (0, 1)^k$ are finite, and the function

$$D_H(\mathbf{y}; r) = y_1 \cdots y_k \exp \left\{ \sum_{i=2}^k (-1)^i \sum_{1 \leq j_1 < \dots < j_i \leq k} d_{J(i)}(\mathbf{y}_{J(i)}) \right\}$$

is a dependence function of H .

(B) The following inequalities hold.

$$D_H(\mathbf{y}; 2r+1) \leq D_H(\mathbf{y}) \leq D_H(\mathbf{y}; 2r),$$

for a nonnegative integer r , where

$$D_H(\mathbf{y}; r) = y_1 \cdots y_k \exp \left\{ \sum_{i=2}^r (-1)^i \sum_{1 \leq j_1 < \dots < j_i \leq k} d_{J(i)}(\mathbf{y}_{J(i)}) \right\}$$

and $D_H(\mathbf{y}; r)$ is a dependence function of a multivariate extreme value distribution.

Proof. It is easily seen that for all $s > 0$,

$$s d_{J(i)}(\mathbf{y}_{J(i)})^{1/s} = d_{J(i)}(\mathbf{y}_{J(i)}).$$

From Theorems 5.3.1 and 5.2.4 of Galambos [1], we have the result. \square

Example 2.1 (See Examples 5.2.2 and 5.2.3 of Galambos [1].) For a Mardia's distribution

$$F(x_1, x_2) = 1 - e^{-x_1} - e^{-x_2} + (e^{x_1} + e^{x_2} - 1)^{-1},$$

$$D_F(y_1, y_2) = y_1 + y_2 - 1 + \left[\frac{1}{1-y_1} + \frac{1}{1-y_2} - 1 \right]^{-1},$$

and

$$\begin{aligned} n \bar{D}_F(y_1^{1/n}, y_2^{1/n}) &= n \left[\frac{1}{1-y_1^{1/n}} + \frac{1}{1-y_2^{1/n}} - 1 \right]^{-1} \\ &\rightarrow - \frac{(\log y_1)(\log y_2)}{\log y_1 + \log y_2}, \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, by Proposition 2.2 we have $F \in D(H)$, where

$$D_H(y_1, y_2) = y_1 y_2 \exp \left[- \frac{(\log y_1)(\log y_2)}{\log y_1 + \log y_2} \right],$$

$$H(x_1, x_2) = \Lambda(x_1) \Lambda(x_2) \exp \{ 1 / (e^{x_1} + e^{x_2}) \},$$

and $\Lambda(x) = \exp(-e^{-x})$.

Proposition 2.3 Let F and G be k -dimensional distributions and let H be a multivariate extreme value distribution.

1) If $F, G \in D(H)$, then

$$\lim_{y \downarrow 0} \frac{1 - D_F(\mathbf{y}')}{1 - D_G(\mathbf{y}')} = 1 \text{ for all } \mathbf{y}' \in (0, 1)^k.$$

2) If $F \in D(H)$, $G_i \in D(H_i)$, $i = 1, \dots, k$, and

$$\lim_{y \uparrow 1} \frac{1 - D_F(\mathbf{y})}{1 - D_G(\mathbf{y})} = 1, \text{ where } \mathbf{1} = (1, \dots, 1),$$

then $G \in D(H)$.

3. Marginally Independent or Perfect Dependent Multivariate Extreme Value Distributions

Let H be a multivariate extreme value distribution with univariate marginals H_i , $i=1, \dots, k$. Let $H_*(x) = H_1(x_1) \cdots H_k(x_k)$ and $H^*(x) = \min\{H_i(x_i), i=1, \dots, k\}$, then it holds

$$H_*(x) \leq H(x) \leq H^*(x)$$

for all $x \in R^k$. Both bounds, H_* and H^* , are multivariate extreme value distributions. Characterizations of these distributions are obtained by Takahashi [5].

In the bivariate case Sibuya [3] obtains necessary and sufficient conditions for $F \in D(H_*)$ and $F \in D(H^*)$. In this section we generalize his results.

Proposition 3.1 *Let F be a k -dimensional distribution and let H_i be a univariate extreme value distribution, $i=1, \dots, k$. Then the following statements are equivalent:*

- 1) $F \in D(H_*)$.
- 2) $F_i \in D(H_i)$, $i=1, \dots, k$, and there exists $y \in (0, 1)^k$ such that

$$\lim_{n \rightarrow \infty} (D_F(y^{1/n}))^n = y_1 y_2 \cdots y_k.$$

- 3) $F_i \in D(H_i)$, $i=1, \dots, k$, and

$$\lim_{y \uparrow 1} \frac{1 - D_F(y \mathbf{1})}{1 - y} = k.$$

- 4) $F_i \in D(H_i)$, $i=1, \dots, k$, and

$$\lim_{y \uparrow 1} \frac{1 - D_F(y \mathbf{1})}{1 - y^k} = 1.$$

Proof. The proof is straightforward from Theorems 2.2 and 4.1 and Corollary 2.4 of Takahashi [6]. \square

Remark. If $k=2$, we have the same result as Proposition 3.1 by Corollary 2.2 of Takahashi [6].

Example 3.1 (See Example 5.2.3 of Galambos [1].) For the Morgenstern distribution

$$F(x_1, x_2) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2} [1 + \alpha(1 - e^{-x_1})(1 - e^{-x_2})],$$

$$D_F(y_1, y_2) = y_1 y_2 [1 + \alpha(1 - y_1)(1 - y_2)]$$

where $-1 \leq \alpha \leq 1$, and

$$\lim_{y \uparrow 1} \frac{1 - D_F(y, y)}{1 - y^2} = 1.$$

By Proposition 3.1 4) we have $F \in D(H_*)$, where $H_*(\cdot, \cdot) = \Lambda(\cdot) \Lambda(\cdot)$.

Proposition 3.2 *Let F be a k -dimensional distribution and let H_i be a univariate extreme value distribution, $i=1, \dots, k$. Then the following statements are equivalent:*

- 1) $F \in D(H^*)$.
- 2) $F_i \in D(H_i)$, $i=1, \dots, k$, and there exists $y \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} (D_F(y^{1/n} \mathbf{1}))^n = y.$$

- 3) $F_i \in D(H_i)$, $i=1, \dots, k$, and

$$\lim_{y \uparrow 1} \frac{1 - D_F(y \mathbf{1})}{1 - y} = 1.$$

Proof. The proof is straightforward from Theorem 3.1 and Corollary 3.1 of Takahashi [6]. \square

4. Joint Asymptotic Distribution of the Multivariate Extreme Statistics

In this section, we show the joint asymptotic distribution of several multivariate extreme statistics along the arguments in Sec. 2.3 of Leadbetter et al. [7]. For simplicity we shall consider the bivariate case.

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a sequence of independent random vectors with common distribution F . The order statistics of the components will be denoted by

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}; \text{ and } Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}.$$

For $i=0, 1, \dots, r-1$, define

$$Z_{n-i} = (X_{n-i:n}, Y_{n-i:n})$$

and let us call Z_{n-i} an $(i+1)$ -th multivariate extreme statistic.

Proposition 4.1 *Suppose that*

$$P\{(Z_n - b_n)/a_n \leq x\} \xrightarrow{w} H(x)$$

for some nondegenerate distribution H . Then, for $x_1 = (x_1, y_1) > x_2 = (x_2, y_2)$,

$$P\{(Z_n - b_n)/a_n \leq x_1, (Z_{n-1} - b_n)/a_n \leq x_2\} \xrightarrow{w} H_J(x_1, x_2)$$

where

$$\begin{aligned} H_J(x_1, x_2) = & H(x_2) \{1 + \log H(x_1) - \log H(x_2) \\ & + [\log H_1(x_1) - \log H_1(x_2) + (h(x_1, y_2) - h(x_2))] \\ & \times [\log H_2(y_1) - \log H_2(y_2) + (h(x_2, y_1) - h(x_2))]\} \end{aligned}$$

and $h(x) = \lim_{n \rightarrow \infty} \bar{F}(a_n x + b_n)$.

Proof. Define

$$\begin{aligned} S_0^n &= \#\{j \mid X_j > a_{1n}x_1 + b_{1n} \text{ or } Y_j > a_{2n}y_1 + b_{2n}, j=1, 2, \dots, n\}, \\ S_1^n &= \#\{j \mid a_{1n}x_2 + b_{1n} < X_j \leq a_{1n}x_1 + b_{1n} \text{ and} \\ &\quad Y_j \leq a_{2n}y_2 + b_{2n}, j=1, 2, \dots, n\}, \\ S_2^n &= \#\{j \mid X_j \leq a_{1n}x_2 + b_{1n} \text{ and} \\ &\quad a_{2n}y_2 + b_{2n} < Y_j \leq a_{2n}y_1 + b_{2n}, j=1, 2, \dots, n\}, \\ S_{12}^n &= \#\{j \mid a_{1n}x_2 + b_{1n} < (X_j, Y_j) \leq a_{1n}x_1 + b_{1n}, j=1, 2, \dots, n\} \end{aligned}$$

then, we have

$$\begin{aligned} P\{(Z_n - b_n)/a_n \leq x_1, (Z_{n-1} - b_n)/a_n \leq x_2\} \\ = P\{S_0^n = 0, S_{12}^n = 0, S_1^n \leq 1, S_2^n \leq 1\} \\ + P\{S_0^n = 0, S_{12}^n = 1, S_1^n = 0, S_2^n = 0\}. \end{aligned}$$

On the other hand, by using Theorem 5.3.1 of Galambos [1], we can evaluate the asymptotic probabilities of the events

$$\{S_0^n = i, S_{12}^n = j, S_1^n = k, S_2^n = m\}$$

for $i, j, k, m = 0, 1$. Thus we have the result. \square

Corollary 4.1 Suppose that

$$P\{(Z_n - b_n)/a_n \leq x\} \xrightarrow{w} H(x)$$

for some nondegenerate distribution H . Then, for fixed $r \geq 1$ and $x_1 > \dots > x_r$

$$\begin{aligned} & \left| P\{(Z_n - b_n)/a_n \leq x_1, \dots, (Z_{n-r+1} - b_n)/a_n \leq x_r\} \right. \\ & \left. - P\{(Z_n^* - \beta_n)/\alpha_n \leq x_1, \dots, (Z_{n-r+1}^* - \beta_n)/\alpha_n \leq x_r\} \right| \rightarrow 0, \\ & \text{as } n \rightarrow \infty, \end{aligned}$$

where Z_{n-i}^* is the $(i+1)$ -th multivariate extreme statistic from the distribution H , $i=0, \dots, r-1$, and $H^n(\alpha_n x + \beta_n) = H(x)$, $n=1, 2, \dots$

Example 4.1 Let F be the bivariate normal distribution with the correlation coefficient less than one. Then

$$\begin{aligned} & \left| P\{(Z_n - b_n)/a_n \leq x_1, \dots, (Z_{n-r+1} - b_n)/a_n \leq x_r\} \right. \\ & \left. - P\{(Z_n^* - (\log n)\mathbf{1}) \leq x_1, \dots, (Z_{n-r+1}^* - (\log n)\mathbf{1}) \leq x_r\} \right| \rightarrow 0, \\ & \text{as } n \rightarrow \infty, \end{aligned}$$

where Z_{n-i}^* is the $(i+1)$ -th multivariate extreme statistic from the bivariate exponential distribution whose marginals are equal to the standard exponential distribution and they are independent. For the univariate case, it is a well known result (see Weissman [8], Theorem 3).

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5. References

- [1] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*. 2nd ed., Krieger, Malabar, FL (1987).
- [2] S. I. Resnick, *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, Berlin (1987).
- [3] M. Sibuya, Bivariate extreme statistics. *Ann. Inst. Statist. Math.* **11**, 195–210 (1960).
- [4] A. W. Marshall and I. Olkin, Domains of attraction of multivariate extreme value distributions. *Ann. Probab.* **11**, 168–177 (1983).
- [5] R. Takahashi, Characterizations of a multivariate extreme value distribution. *Adv. Appl. Probab.* **20**, 235–236 (1988).
- [6] R. Takahashi, Asymptotic independence and perfect dependence of vector components of multivariate extreme statistics. *Statist. Probab. Letters* **19**, 19–26 (1994).
- [7] M. R. Leadbetter, G. Lindgren, and H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*. Springer-Verlag, New York (1983).
- [8] I. Weissman, Estimation of parameters and large quantiles based on the k largest observations. *J. Amer. Statist. Assoc.* **73**, 812–815 (1978).

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